ON THE GAMMA FACTORS ATTACHED TO REPRESENTATIONS OF $U(2,1)$ OVER A p-ADIC FIELD

BY

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ABSTRACT

In this paper we complete the local non-archimedean theory of Rankin–Selberg convolutions for $U(2,1)$ that was suggested by S. Gelbart and I. Piatetski-Shapiro in [G,PS,1]. In addition, we prove two fundamental properties of the gamma factors (Theorem 6.3) which allow us to give a new proof of a strong multiplicity one theorem for $U(2,1)$.

Introduction

Let F be a p-adic field and E a quadratic extension of F. Given a generic representation π of $U(2,1)(F)$ and a quasi-character χ of E^*, Gelbart and Piatetski-Shapiro [G,PS,1] introduced a family of Zeta integrals which interpolate a degree 6 L-function over F. In this paper we prove the absolute convergence of these integrals in a right half plane and we show that they are rational functions in q_F^s where q_F is the order of the residue field of F (section 3). We prove the existence of a local gamma and L-factors (section 4). We end the local theory with our main result, Theorem 6.3. We show that a generic representation of $U(2,1)$ is determined by its central character and gamma factors. We also show that two generic representations will have the same gamma factors if the quasi character χ is highly ramified (sections 5 and 6). We end with a global application which is a new proof of a strong multiplicity one result due to Gelbart, Rogawski

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and Soudry [Rog], [G,R,S]. We also give the global functional equation which is implicit in [G,PS,1].

Most of the ideas here are inspired by [J], [G,PS,2], [S] and [H]. The second part of our main result is inspired by a similar unpublished result of Cogdell and Piatetski-Shapiro [C,PS]. I would like to thank Jim Cogdell for showing me his notes.

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1. Notations and preliminaries

Let F be a nonarchimedean local field of characteristic 0. Let E be a quadratic extension of F and set $E = F[\sqrt{\epsilon}]$, where ϵ is a nonsquare in F. Let $x \to \bar{x}$ be the Galois automorphism of E over F sending $\sqrt{\epsilon}$ to $-\sqrt{\epsilon}$. For a local field L, we let R_L and P_L be the ring of integers and maximal ideal in L respectively. Let ϖ_L be a generator of P_L. We denote by $|x|_L$ the normalized absolute value of $x \in L$, and we let q_L be the order of the residue class field of L. Let χ be a quasi-character of L^*. We say that χ is ramified of degree d if $\chi(1+P_L^d) = 1$ and $\chi(1+P_L^{d-1}) \neq 1$. We set $l(\chi) = d$. We say that χ is highly ramified if $l(\chi) >> 0$. Whenever we do not mention the local field then it is assumed to be E. For example, if $x \in F$ then $|x| = |x|_E$. Let

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

We define

$$G = U(2,1)(F) = \{A \in \mathrm{GL}_3(E) \| A^* J A = J\}$$

where $A^* = (\bar{A})^t$.

We will consider the following subgroups of G. Let C be the center of G, B be the upper triangular Borel subgroup of G and U the unipotent radical of B. Then

$$U = \left\{ u(y,z) = \begin{pmatrix} 1 & y & z \\ & 1 & -\bar{y} \\ & & 1 \end{pmatrix} \|u(y,z) \in G \right\}.$$

Let

$$H = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in G \right\}, \quad B_H = \left\{ \begin{pmatrix} a & 0 & r \\ & 1 & 0 \\ & & \bar{a}^{-1} \end{pmatrix} \in H \right\},$$

$$U_H = \left\{ x(r) = \begin{pmatrix} 1 & 0 & r \\ & 1 & 0 \\ & & 1 \end{pmatrix} \| r \in E, r + \bar{r} = 0 \right\},$$

$$\hat{U}_H = \left\{ \hat{x}(r) = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ r & 0 & 1 \end{pmatrix} \| r \in E, r + \bar{r} = 0 \right\},$$

$$T_H = \left\{ t(a) = \begin{pmatrix} a & & \\ & 1 & \\ & & \bar{a}^{-1} \end{pmatrix} \| a \in E^* \right\}.$$

Let $K = \mathrm{GL}_3(R_E) \bigcap G$ be a maximal compact subgroup of G and $K_H = K \bigcap H$ be a maximal compact subgroup of H. For a subgroup Q of $\mathrm{GL}_2(E)$ we denote

$$m(Q) = \left\{ m(q) = \begin{pmatrix} q_{1,1} & 0 & q_{1,2} \\ 0 & 1 & 0 \\ q_{2,1} & 0 & q_{2,2} \end{pmatrix} \| q = (q_{i,j}) \in Q \right\}.$$

Thus $H = m(U_{1,1})$.

Let $S(H) = \{A \in H \| \det(A) = 1\}$.

Let $d(a, b) =$ diagonal (a, b) and $d(a) = d(a, 1)$.

Then $S(H) \cong \mathrm{SU}_{1,1}(F) \cong \mathrm{SL}_2(F)$ and this isomorphism can be realized through the following equality:

$$S(H) = m(d(\sqrt{\epsilon})) \cdot m(\mathrm{SL}_2(F)) \cdot m(d(1/\sqrt{\epsilon})).$$

Let ψ_F be a a nontrivial character of F and ψ_E be a nontrivial character of E. (In the global applications ψ_F and ψ_E will be related by the formula

$$\psi_E(x) = \psi_F\left(\frac{x - \bar{x}}{2\sqrt{\epsilon}}\right)$$

(see [G,PS,1]). It will be convenient for us to use this slightly more general situation.) We define a nondegenerate character ψ on U by

$$\psi(u(y, z)) = \psi_E(y).$$

Let (π, V) be an irreducible admissible representation of G. We say that π is generic if there exists a nontrivial functional $l\colon V \to C$ such that:

$$l(\pi(u)v) = \psi(u)l(v), \quad u \in U, \quad v \in V.$$

It is well known that such a functional is unique up to scalar multiples. We call this functional the Whittaker functional. Now define

$$W_v(g) = l(\pi(g)(v)), \quad v \in V, \quad g \in G$$

and let G act on the space of these functions by right translations. Then the map $v \to W_v$ gives a realization of π on a space of functions satisfying

$$W_v(ug) = \psi(u)W_v(g), \quad u \in U, \quad g \in G.$$

We denote this space by $\mathcal{W}(\pi, \psi)$.

For a p-adic group G, a subgroup H and a character χ of H, we let $\mathrm{Ind}_H^G \chi$ be the nonnormalized induced representation whose underlying space is the set of smooth functions $f\colon G \to \mathbf{C}$ satisfying

$$f(hg) = \chi(h)f(g) \quad \text{for all } h \in H, \quad g \in G.$$

We let G act on this space by right translations,

$$(\rho(g')f)(g) = f(gg'), g, g' \in G.$$

2. Zeta integrals

In this section we define some zeta integrals which are the main study of this paper. The definition of these integrals is sketched in [G,PS,1].

Let $S(F^2)$ be the space of locally constant, compactly supported functions on F^2. For $\Phi \in S(F^2)$ we define the Fourier transform

$$\hat{\Phi}(x, y) = \int \Phi(u, v)\psi_F(yu - xv)dudv.$$

Let $g \in \mathrm{GL}_2(F)$. Set $(g\Phi)(x, y) = \Phi[(x, y)g]$. Then

$$(g\Phi)\hat{} = |\det(g)|_F^{-1} g'\hat{\Phi} \tag{2.1}$$

where

$$g' = \begin{pmatrix} \det(g)^{-1} & \\ & \det(g)^{-1} \end{pmatrix} g.$$

Let $\Phi \in S(F^2)$, $s \in \mathbf{C}$ and χ be a quasi-character of E^*. Let $g \in \mathrm{GL}_2(F)$. As in ([Go,J], §14) we set

$$z(s, g, \Phi, \chi) = \int_{F^*} (g\Phi)(0, r)\chi(r)|r|_E^s d^*r. \tag{2.2}$$

Notice that here χ is viewed as a quasi-character of F^* by looking at the restriction to F^*. The following Lemma is proved in [Go,J]:

LEMMA 2.3: *For a fixed χ, $z(s, g, \Phi, \chi)$ is defined by an absolutely convergent integral for* Re(s) *large enough, and by meromorphic continuation it can be defined for almost all s.*

Using the Bruhat decomposition for H (or Hilbert's Satz 90), it is easy to see that each $h \in H$ can be written (not uniquely) in the form

$$h = t(a)\tilde{h}, \quad a \in E^*, \quad \tilde{h} \in S(H).$$

Since $\tilde{h} \in S(H)$ we can write $\tilde{h} = m(d(\sqrt{\epsilon}))\cdot m(h_1)\cdot m(d(1/\sqrt{\epsilon}))$ for $h_1 \in \mathrm{SL}_2(F)$, and we get

$$h = t(a)\cdot m(d(\sqrt{\epsilon}))\cdot m(h_1)\cdot m(d(1/\sqrt{\epsilon})), a \in E^*, h_1 \in \mathrm{SL}_2(F). \tag{2.4}$$

For $h \in H$ written in the form (2.4) we set

$$f(s, h, \Phi, \chi) = \chi(a)|a|^s z(s, h_1\Phi, \chi).$$

LEMMA 2.5: *The definition of $f(s, h, \Phi, \chi)$ is independent of the choice of decomposition* (2.4) *of $h \in H$.*

Proof: Choose two decompositions

$$h = t(a)\cdot m(d(\sqrt{\epsilon}))\cdot m(h_1)\cdot m(d(1/\sqrt{\epsilon})) = t(b)\cdot m(d(\sqrt{\epsilon}))\cdot m(h_1')\cdot m(d(1/\sqrt{\epsilon})).$$

We have $h_1 = d(a^{-1}b, \bar{a}\bar{b}^{-1})h_1'$, hence $a^{-1}b \in F$, hence $\bar{a}^{-1}\bar{b} = a^{-1}b$. Thus

$$\begin{aligned}
&\chi(a)|a|_E^s \int \Phi[(0, r)h_1]\chi(r)|r|_E^s d^*r \\
=&\chi(a)|a|_E^s \int \Phi[(0, t)d(a^{-1}b, \bar{a}\bar{b}^{-1})h_1']\chi(r)|r|_E^s d^*r \\
=&\chi(b)|b|_E^s \int \Phi[(0, r)h_1']\chi(r)|r|_E^s d^*r. \qquad \blacksquare
\end{aligned}$$

It is easy to see that the function f defined by $f(h) = f(s,h,\Phi,\chi)$ is an element of $\mathrm{Ind}_{B_H}^{H}\chi|\,|^s$, where

$$\chi|\,|^s(t(a)u) = \chi(a)|a|^s, \quad a \in E^*, \quad u \in U_H.$$

LEMMA 2.6: *Let $y, h \in H$ and write*

$$y = t(b)m(d(\sqrt{\epsilon}))m(y_1)m(d(1/\sqrt{\epsilon})), \quad h = t(a)m(d(\sqrt{\epsilon}))m(h_1)m(d(1/\sqrt{\epsilon}))$$

where $a, b \in E^*$, $y_1, h_1 \in \mathrm{SL}_2(F)$ as in (2.4). Then

$$f(s, yh, \Phi, \chi) = \chi(a)|a|^s f(s, y, d(a\bar{a})h_1\Phi, \chi).$$

Proof:

$$\begin{aligned} f(s,yh,\Phi,\chi) &= \chi(b)|b|^s\chi(a)|a|^s \int [(t(a^{-1})y_1t(a)h_1\Phi)(0,r)]\chi(r)|r|_E^s\, d^*r \\ &= \chi(b)|b|^s\chi(a)|a|^s \int [(y_1d(a\bar{a})h_1\Phi)(0,r)]\chi(r)|r|_E^s\, d^*r \\ &= \chi(a)|a|^s f(s,y,d(a\bar{a})h_1\Phi,\chi). \qquad ∎ \end{aligned}$$

We are now in the position to define our main objects, the zeta integrals. (See [G,PS,1] (4.1).)

Let (π, V) be a generic representation of G with a Whittaker model $\mathcal{W}(\pi,\psi)$. For any $W \in \mathcal{W}(\pi,\psi)$ we define

$$Z(s,W,\Phi,\chi) = \int_{U_H \setminus H} W(h)f(s,h,\Phi,\chi)dh. \tag{2.7}$$

We shall address the convergence of these zeta integrals later. We shall also show that there is a functional equation involving

$$A(s,W,\Phi,\chi) = Z(s,W,\Phi,\chi) \quad \text{and} \quad \tilde{A}(s,W,\Phi,\chi) = Z(1-s,W,\hat{\Phi},\bar{\chi}^{-1})$$

where $\bar{\chi}(x) = \chi(\bar{x})$. For now, let us look at the invariance properties of these integrals.

LEMMA 2.8: *Let $h \in H$, write*

$$h = t(a)m(d(\sqrt{\epsilon}))\cdot m(h_1)\cdot m(d(1/\sqrt{\epsilon})), \quad a \in E^*, \quad h_1 \in \mathrm{SL}_2(F) \quad (\textit{see } (2.4)).$$

Then

$$A(s,\pi(h)W,d(a\bar{a})h_1\Phi,\chi)=\chi^{-1}(a)|a|^{-s}A(s,W,\Phi,\chi),$$
$$\tilde{A}(s,\pi(h)W,d(a\bar{a})h_1\Phi,\chi)=\chi^{-1}(a)|a|^{-s}\tilde{A}(s,W,\Phi,\chi).$$

Proof: Using Lemma 2.6 we can see that

$$\begin{aligned}A(s,\pi(h)W,d(a\bar{a})h_1\Phi,\chi)&=\int_{U_H\setminus H}W(yh)f(s,y,d(a\bar{a})h_1\Phi,\chi)dy\\&=\chi^{-1}(a)|a|^{-s}\int_{U_H\setminus H}W(yh)f(s,yh,\Phi,\chi)dy\\&=\chi^{-1}(a)|a|^{-s}A(s,W,\Phi,\chi).\end{aligned}$$

To prove the second equality we use (2.1) and Lemma 2.6 to see that

$$\begin{aligned}f(1-s,y,\,(d(a\bar{a})h_1\Phi)\hat{},\bar{\chi}^{-1})&=\chi^{-1}(a\bar{a})|a|^{1-2s}f(1-s,y,d(a\bar{a})h_1\hat{\Phi},\bar{\chi}^{-1})\\&=\chi^{-1}(a)|a|^{-s}f(1-s,yh,\hat{\Phi},\bar{\chi}^{-1}).\end{aligned}$$

Hence

$$\begin{aligned}\tilde{A}(s,\pi(h)W,d(a\bar{a})h_1\Phi,\bar{\chi}^{-1})&=\int_{U_H\setminus H}W(yh)f(1-s,y,(d(a\bar{a})h_1\Phi)\hat{},\bar{\chi}^{-1})dy\\&=\chi^{-1}(a)|a|^{-s}\int_{U_H\setminus H}W(yh)f(1-s,yh,\hat{\Phi},\bar{\chi}^{-1}dy\\&=\chi^{-1}(a)|a|^{-s}\tilde{A}(s,W,\Phi,\chi).\end{aligned}$$ ∎

3. Convergence of the integrals

Let W be a Whittaker function on G. Thus $W\colon G\to\mathbf{C}$ is smooth on the right and satisfies

$$W(ug)=\psi(u)W(g),\quad u\in U,\quad g\in G.$$

LEMMA 3.1: *The function $a\to W(t(a))$ has bounded support in E.*

Proof: Choose $y\in E$ with $|y|$ small enough, so that $W(gu(y,-y\bar{y}/2))=W(g)$ for all $g\in G$. We have

$$\begin{aligned}W(t(a))=W(t(a)u(y,-y\bar{y}/2))&=W(t(a)u(y,-y\bar{y}/2)t(a)^{-1}t(a))\\&=\psi(ay)W(t(a)).\end{aligned}$$

Thus, for a to be in the support, we must have $\psi(ay)=1$ for all small y. This implies that $|a|$ is bounded. ∎

Let π be an irreducible admissible representation of G acting by right translations on a space of Whittaker functions $\mathcal{W}(\pi,\psi)$. We have the following analogue to Proposition 2.2 in [J,PS,S 1].

PROPOSITION 3.2: *There is a finite set A of finite functions on E^*, such that for any W in $\mathcal{W}(\pi,\psi)$ there exist, for every $\alpha \in A$, Schwartz functions $\phi_\alpha \in S(F)$ satisfying*

$$W(t(a)) = \sum_{\alpha \in A} \phi_\alpha(a)\alpha(a), \quad a \in E^*.$$

Proof: The proof is a repetition of that of Proposition 2.2 in [J,PS,S 1]. Let

$$\mathcal{W}' = \mathrm{Span}\{\pi(u)W - W \colon u \in U, W \in \mathcal{W}(\pi,\psi)\}.$$

As in the proof of Lemma 3.1, we see that for $W \in \mathcal{W}'$, $W(t(a)) = 0$ for $|a|$ small (and for $|a|$ large). Since the representation π' of B on $\mathcal{W}(\pi,\psi)/\mathcal{W}'$ is finite dimensional we have that $\mathrm{Span}\{\pi'(t(a))\colon a \in E^*\}$ is finite dimensional. Now we are at the situation of the end of the proof of Proposition 2.2 in [J,PS,S 1]. ∎

We define a gauge on G to be a function ξ of the form

$$\xi(u \cdot c \cdot t(a) \cdot k) = \alpha(a)\phi(a)$$

where $a \in E^*$, $u \in U$, $c \in C$ (the center of G), $k \in K$ (the standard maximal compact of G), α is a sum of positive characters and ϕ is a nonnegative element of $S(E)$. Clearly a sum of gauges is majorized by a gauge and since the center of G is compact, Proposition 3.2 implies (as in Proposition 2.3.5 in [J,PS,S 1])

PROPOSITION 3.3: *For every $W \in \mathcal{W}(\pi,\psi)$ there is a gauge ξ on G such that*

$$|W(g)| \leq \xi(g), \quad g \in G.$$

PROPOSITION 3.4: *Let χ be a quasi-character of E^*. There is $s_0 \in \mathbf{R}$ such that the integral*

$$Z(s, W, \Phi, \chi) = \int_{U_H \setminus H} W(h) f(s, h, \Phi, \chi) dh, \quad W \in \mathcal{W}(\pi,\psi), \quad \Phi \in S(F)$$

converges absolutely for $\mathrm{Re}(s) > s_0$. Moreover, as a function of s it is a rational function in q_F^{-s} and hence can be meromorphically continued to the whole plane.

Proof: Let K_H be the maximal compact of H and we use the Iwasawa decomposition $H = U_H T_H K_H$ to compute our integral. We have

$$Z(s, W, \Phi, \chi) = \int_{E^*} \int_{K_H} W(t(a)k) f(s, t(a)k, \Phi, \chi) d^*a dk.$$

Using a gauge for W as in Proposition 3.3 and the fact that

$$f(s,t(a)k,\Phi,\chi)=\chi(a)|a|^s f(s,k,\Phi,\chi),\quad a\in E^*,\quad k\in K_H,$$

we can see that this integral is majorized by a sum of integrals of the form

$$\int_{E*}\alpha(a)\chi(a)\phi(a)|a|^s d^*a \tag{3.5}$$

where $0\le\phi\in S(E)$ and α is a sum of positive characters. This integral converges for $\mathrm{Re}(s)$ large (depending only on the characters which decompose α and on χ). Furthermore, we can see that if we use Proposition 3.2 instead of Proposition 3.3 we get that $Z(s,W,\Phi,\chi)$ is a finite linear combination of integrals of the form (3.5) where $\phi\in S(E)$, not necessarily positive and α a finite function on E^*. The integral (3.5) in this case is easily seen to be a rational function in q_F^s when it converges. ∎

4. The functional equation

In this section we prove that the space of bilinear forms satisfying the equivariance properties of Lemma 2.7 is in general one dimensional. We follow here the methods of [J] and [G,PS,2].

Consider the subgroup $B_0=\{t(a)u\|a\in E,u\in U\}$ of B. U_H, the center of U, is a normal subgroup in B, hence in B_0, and we have

$$U_H\diagdown B_0\cong P_2=P_2(E)$$

where

$$P_2=\left\{\begin{pmatrix}a & b\\ & 1\end{pmatrix}\|a\in E^*,b\in E\right\}.$$

Let $Z_2\subset P_2$ be the subgroup

$$Z_2=\left\{z(b)=\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\|b\in E\right\}.$$

Since ψ is trivial on U_H, ψ induces a character on $U_H\diagdown U\cong Z_2$ which we again denote by ψ. This character is given by

$$\psi(z(b))=\psi_E(b).$$

Let B_{SL} be the Borel subgroup of $\mathrm{SL}_2(F)$ and let χ be a quasicharacter of E^*. Let $\chi||^s$ be a quasi-character of B_{SL} or of B_H defined by

$$\chi||^s(b) = \chi(b_{1,1})|b_{1,1}|_E^s, \quad b = (b_{i,j}).$$

For every $f \in \mathrm{Ind}_{B_{\mathrm{SL}}}^{\mathrm{SL}_2(F)} \chi||^s$, we define $\tilde{f} \in \mathrm{Ind}_{B_H}^{H} \chi||^s$ by

$$\tilde{f}(h) = \chi(a)|a|^s f(h_1)$$

where $h \in H$ and $h = t(a) \cdot m(d(\sqrt{\epsilon}) \cdot m(h_1) \cdot m(d(1/\sqrt{\epsilon})), a \in E^*, h_1 \in \mathrm{SL}_2(F)$, as in (2.4). It is easy to see that $\tilde{f}$ is well defined. Moreover, if $p \in \mathrm{Ind}_{B_H}^{H} \chi||^s$ we can find an $f \in \mathrm{Ind}_{B_{\mathrm{SL}}}^{\mathrm{SL}_2(F)} \chi||^s$ such that $p = \tilde{f}$. To do that we define

$$f(g) = p(m(d(\sqrt{\epsilon}^{-1})) \cdot m(g) \cdot m(d(\sqrt{\epsilon}))), \quad g \in \mathrm{SL}_2(F).$$

Hence we have

LEMMA 4.1: *The mapping $f \to \tilde{f}$ gives a bijection between $\mathrm{Ind}_{B_{\mathrm{SL}}}^{\mathrm{SL}_2(F)} \chi||^s$ and $\mathrm{Ind}_{B_H}^{H} \chi||^s$.*

LEMMA 4.2: *There is $s_0 \in \mathbf{R}$ such that for every s with $\mathrm{Re}(s) > s_0$ and every $f \in \mathrm{Ind}_{B_H}^{H} \chi||^s$ there exist $\Phi \in S(F^2)$ such that*

$$f(s, h, \Phi, \chi) = f(h) \quad \textit{for all } h \in H.$$

Proof: It follows from the proof of Proposition 3.2 of [J,L] that if $\mathrm{Re}(s)$ is large enough, then every $f \in \mathrm{Ind}_{B_{\mathrm{SL}}}^{\mathrm{SL}_2(F)} \chi||^s$ is of the form

$$f(h) = \int (h\Phi)(0, r)\chi(r)|r|_E^s d^*r$$

for some $\Phi \in S(F^2)$. Our Lemma now follows from Lemma 4.1. ∎

LEMMA 4.3 ([J] Lemma 14.7.1): *There is s_0 in $\mathbf{R}$ so that for all s with $\mathrm{Re}(s) > s_0$ and all $\Phi \in S(F^2)$, the relation*

$$z(s, g, \Phi, \chi) = 0 \quad \textit{for all } g \in \mathrm{GL}_2(F)$$

implies the relation

$$z(1 - s, g, \hat{\Phi}, \chi^{-1}) = 0 \quad \textit{for all } g \in \mathrm{GL}_2(F).$$

It is clear from the proof of Lemma 14.7.1 in [J] that the same holds if we replace $\mathrm{GL}_2(F)$ with $\mathrm{SL}_2(F)$ in the statement of Lemma 4.3. Thus we get the following Corollary.

LEMMA 4.4: *There is $s_0 \in \mathbf{R}$ so that for all s with $\mathrm{Re}(s) > s_0$, if $f(s,h,\Phi,\chi) = 0$ for all $h \in H$ then $f(1-s,h,\hat{\Phi},\bar{\chi}^{-1}) = 0$ for all $h \in H$.*

Let (π, V) be a generic representation of G with Whittaker model $\mathcal{W}(\pi,\psi)$. Combining Lemma 4.2, Lemma 4.4 and Lemma 2.8, we get

LEMMA 4.5: *There is $s_0 \in \mathbf{R}$ so that for $\mathrm{Re}(s) > s_0$, there are unique bilinear forms β_s and β'_s on $\mathcal{W}(\pi,\psi) \times \mathrm{Ind}_{B_H}^H \chi||^s$, such that if f is the function defined by $f(h) = f(s,h,\Phi,\chi)$, we have*

$$\beta_s(W,f) = Z(s,W,\Phi,\chi) \quad \text{for all } W \in \mathcal{W}(\pi,\psi) \text{ and all } \Phi \in S(F^2)$$

and

$$\beta'_s(W,f) = Z(1-s,W,\hat{\Phi},\bar{\chi}^{-1}) \quad \text{for all } W \in \mathcal{W}(\pi,\psi) \text{ and all } \Phi \in S(F^2).$$

Moreover, it follows from Lemma 2.8 that β_s and β'_s satisfy the following invariance property.

$$\beta_s(\pi(h)W,\rho(h)f) = \beta_s(W,f), \quad \text{for all } W \in \mathcal{W}(\pi,\psi), \quad f \in \mathrm{Ind}_{B_H}^H \chi||^s, \quad h \in H \tag{4.6}$$

where $(\rho(h)f)(g) = f(gh)$. By analytic continuation, this formula is true for almost all s.

We are now in the position to prove our uniqueness Theorem. (See Theorem 7.1 of [G,PS,2].)

THEOREM 4.7: *Let $\mathcal{B}_s$ be the space of bilinear forms on $\mathcal{W}(\pi,\psi) \times \mathrm{Ind}_{B_H}^H \chi||^s$ satisfying (4.6). Then outside a finite number of values of q^s, $\mathcal{B}_s$ is one dimensional.*

Proof: Recall that for any pair of smooth representations π_1, π_2 of a p-adic group H, the vector space of H-invariant forms on $V_{\pi_1} \times V_{\pi_2}$ is isomorphic to $\mathrm{Hom}_H(\pi_1, \tilde{\pi}_2)$, where $\tilde{\pi}_2$ is the contragredient representation to π_2. Hence

$$\mathcal{B}_s \cong \mathrm{Hom}_H(V, \mathrm{Ind}_{B_H}^H \chi^{-1}||^{1-s}),$$

where $V = V_\pi$. By the Frobenius reciprocity theorem

$$\mathrm{Hom}_H(V, \mathrm{Ind}_{B_H}^H \chi^{-1}||^{1-s}) \cong \mathrm{Hom}_{T_H}(V^{U_H}, \chi^{-1}||^{1-s}),$$

where V^{U_H} is the Jacquet module, $V^{U_H} = V/V(U_H)$, and

$$V(U_H) = \mathrm{Span}\{\pi(x)v - v \| v \in V, x \in U_H\}.$$

Since B_0 normalizes U_H and $U_H \setminus B_0 \cong P_2$, we have that P_2 acts on V^{U_H}. Let

$$V^{U_H}(Z_2) = \mathrm{Span}\{\pi(z)v - v \| v \in V^{U_H}, z \in Z_2\}.$$

We claim that $V^{U_H,Z_2} = V^{U_H}/V^{U_H}(Z_2)$ is a finite-dimensional P_2 module and that $V^{U_H}(Z_2)$ is irreducible as a P_2 module and is isomorphic to $\mathrm{ind}_{Z_2}^{P_2}\psi$. (Here ind is the compactly supported induction.)

Since $V^{U_H}/V^{U_H}(Z_2) \cong V^U$ where V^U is the standard Jacquet module of V with respect to the unipotent radical U, we immediately get that V^{U_H,Z_2} is finite dimensional. To prove the second part of the claim we first look at the twisted Jacquet functor used in [BZ]. Let

$$V^{U_H}(Z_2,\psi) = \mathrm{Span}\{\pi(z)v - \psi(z)v \| v \in V^{U_H}, z \in Z_2\}.$$

Let $D = V^{U_H}/V^{U_H}(Z_2,\psi)$. D is a Z_2 module with the action $\pi(z)d = \psi(z)d$ for all $d \in D$ and $z \in Z_2$. We claim that D is one dimensional. Since the Whittaker functional vanishes on $V(U_H)$, it defines a nonzero functional on V^{U_H}. Since this functional vanishes on $V^{U_H}(Z_2,\psi)$, it defines a nonzero functional on D, hence $D \neq 0$. Now assume that $\dim(D) > 1$. Then there exist two linearly independent functionals l_1 and l_2 on D. l_1 and l_2 induce two independent Whittaker functionals on V which is a contradiction to π being irreducible. Hence D is one dimensional. By the theory of Bernstein and Zelevinski [BZ] there is an exact sequence of P_2 modules

$$0 \to \mathrm{ind}_{Z_2}^{P_2} D \to V^{U_H} \to V^{U_H} \setminus V^{U_H}(Z_2) \to 0$$

where the third arrow is the natural map to the quotient space. Hence $V^{U_H}(Z_2) \cong \mathrm{ind}_{Z_2}^{P_2} D$ as a P_2 module, and our claim is proved.

We now would like to show that $\mathrm{Hom}_{T_H}(V^{U_H}, \chi^{-1}||^{1-s})$ is one dimemsional outside a finite number of values of q^s. By [J,L] we know that

$$\dim(\mathrm{Hom}_{T_H}(\mathrm{Ind}_{Z_2}^{P_2}\psi, \chi^{-1}||^{1-s})) = 1$$

for all ramified quasi-characters χ of E^*. If χ is unramified then the dimension is one for all but one value of q^s. Let $\dim(V^{U_H,Z_2}) = r$. There exist r quasi-characters of E^*, $\chi_1, \ldots, \chi_r$, and a sequence of T_H submodules of V^{U_H}

$$V^{U_H}(Z_2) = W_0 \subset W_1 \subset \cdots \subset W_r = V^{U_H}$$

such that $W_i \setminus W_{i-1}$, $i = 1, \ldots, r$, is one dimensional and that T_H acts on $W_i \setminus W_{i-1}$ as the character χ_i, $i = 1, \ldots, r$. We claim that if s is such that $\chi^{-1}||^{1-s} \neq \chi_i$, $i = 1, \ldots, r$, then given an element of $\mathrm{Hom}_{T_H}(V^{U_H}(Z_2), \bar{\chi}||^s)$ there is at most one way to extend it to an element of $\mathrm{Hom}_{T_H}(V^{U_H}, \bar{\chi}||^s)$.

To show that, let $L \in \mathrm{Hom}_{T_H}(W_{i-1}, \chi^{-1}||^{1-s})$ for some fixed i, $0 \leq i < r$. Let L' be an extension of L to $\mathrm{Hom}_{T_H}(W_i, \chi^{-1}||^{1-s})$. Let $v \in W_i$, $v \notin W_{i-1}$. Since W_i/W_{i+1} is one dimensional and T_H acts on this quotient with the character χ_i we have that

$$\pi(t)v = \chi_i(t)v + w_t, \qquad \text{for all } t \in T_H$$

where $w_t \in W_{i-1}$. Hence

$$L'(\pi(t)v) = \chi^{-1}||^{1-s}(t)L'(v) = \chi_i(t)L'(v) + L(w_t)$$

so $L'(v)$ is determined by L if $\chi^{-1}||^{1-s} \neq \chi_i$.

Thus, for such s for which $\chi^{-1}||^s \neq \chi_i, i = 1, \ldots, r$, we have that $\mathrm{Hom}_{T_H}(V^{U_H}, \chi^{-1}||^{1-s})$ is at most one dimensional. It is also possible to show that we can always extend an element of $\mathrm{Hom}_{T_H}(V^{U_H}(Z_2), \chi^{-1}||^{1-s})$ to an element of $\mathrm{Hom}_{T_H}(V^{U_H}, \chi^{-1}||^{1-s})$. However, we will derive our Theorem by showing that our zeta integrals do not vanish identically. We postpone that to Section 6 (cf. (6.5)). ∎

COROLLARY 4.8: *There is a rational function in* q_F^{-s}, $\gamma(s, \pi, \chi, \psi_F, \psi_E)$ *such that*

$$\gamma(s, \pi, \chi, \psi_F, \psi_E) Z(s, W, \Phi, \chi) = Z(1 - s, W, \hat{\Phi}, \bar{\chi}^{-1}).$$

When ψ_F and ψ_E are linked, as in the global application, we shall denote it by

$$\gamma(s, \pi, \chi, \psi) = \gamma(s, \pi, \chi, \psi_F, \psi_E).$$

We will later use a particular choice for ψ_E and ψ_F. Thus, it is important for us to know how the γ-factor $\gamma(s, \pi, \chi, \psi_F, \psi_E)$ changes when we change ψ_F and ψ_E.

LEMMA 4.9: *Let $\delta \in E^*$ and let $\psi'_E(x) = \psi_E(\delta x)$. Then*

$$\gamma(s,\pi,\chi,\psi_F,\psi'_E) = \chi(\delta\bar{\delta}^{-1})|\delta|^{2s-1}\gamma(s,\pi,\chi,\psi_F,\psi_E).$$

Proof: The Whittaker model of (π, V) with respect to the character ψ'_E is given by

$$v \to W_v^{t(\delta)}$$

where $W_v^{t(\delta)}(g) = W_v(t(\delta)g)$ for all $g \in G$. We have

$$\begin{aligned} Z(s, W^{t(\delta)}, \Phi, \chi) &= \int_{U_H \smallsetminus H} W(t(\delta)h) f(s,h,\Phi,\chi)dh \\ &= \int_{U_H \smallsetminus H} W(h) f(s, t(\delta^{-1})h, \Phi, \chi)dh \\ &= \chi^{-1}(\delta)|\delta|^{-s} Z(s, W, \Phi, \chi). \end{aligned}$$

Similarly, we have

$$Z(1-s, W^{t(\delta)}, \hat{\Phi}, \bar{\chi}^{-1}) = \chi(\bar{\delta})|\delta|^{s-1} Z(1-s, W, \hat{\Phi}, \bar{\chi}^{-1}).$$

Using the functional equations for $\gamma(s,\pi,\chi,\psi_F,\psi_E)$ and $\gamma(s,\pi,\chi,\psi_F,\psi'_E)$ we get our result. ∎

LEMMA 4.10: *Let $\delta \in F^*$ and let $\psi'_F(x) = \psi_F(\delta x)$. Then*

$$\gamma(s,\pi,\chi,\psi'_F,\psi_E) = \chi(\delta)|\delta|^{s-1}\gamma(s,\pi,\chi,\psi_F,\psi_E).$$

Proof: For $\Phi \in S(F^2)$ we define another Fourier transform

$$\tilde{\Phi}(x,y) = \int \Phi(u,v)\psi'_F(yu - xv)dudv.$$

We have that $\tilde{\Phi} = \mathrm{diag}(\delta,\delta)\hat{\Phi}$, hence

$$f(1-s,h,\hat{\Phi},\chi) = \chi(\delta)|\delta|^{s-1} f(1-s,h,\tilde{\Phi},\chi).$$

Using the functional equations for $\gamma(s,\pi,\chi,\psi_F,\psi_E)$ and $\gamma(s,\pi,\chi,\psi'_F,\psi_E)$ we get our result. ∎

It follows from Lemma 4.9 that the set of poles of the rational functions $Z(s, W, \Phi, \chi)$ is independent of the choice of ψ_E. Moreover, when we vary the character ψ_E and look at the subspace $I_{\pi,\chi}$ of $\mathbf{C}(q^{-s})$ defined by

$$I_{\pi,\chi} = \mathrm{Span}\{Z(s,W,\Phi,\chi) \| W \in \mathcal{W}(\pi,\psi), \psi \text{ nontrivial}\}$$

we can see that $I_{\pi,\chi}$ is closed under multiplication by q^s and q^{-s}, hence is a fractional ideal. It will follow from (6.5) that $I_{\pi,\chi}$ is generated by a function of the form $1/P(q^{-s})$ where $P(x)$ is a polynomial such that $P(0) = 1$. We define the L-factor $L(s,\pi,\chi)$ to be

$$L(s,\pi,\chi) = \frac{1}{P(q^{-s})}. \tag{4.11}$$

We define the ϵ-factor $\epsilon(s,\pi,\chi,\psi_F,\psi_E)$ to be

$$\epsilon(s,\pi,\chi,\psi_F,\psi_E) = \gamma(s,\pi,\chi,\psi_F,\psi_E)\frac{L(s,\pi,\chi)}{L(1-s,\tilde{\pi},\bar{\chi}^{-1})}. \tag{4.12}$$

5. Howe vectors

In this section we attach to each generic representation (π, V) of G a sequence of vectors $v_m \in V$, and study their properties. (See [H].)

From now on we shall assume that $\psi_F(R_F) = 1$ and $\psi_F(\varpi_F^{-1}R_F) \neq 1$ and that $\psi_E(R_E) = 1$ and $\psi_E(\varpi_E^{-1}R_E) \neq 1$.

Let $\tilde{K}_m \subset \mathrm{GL}_3(E)$ be a congruence subgroup defined by $\tilde{K}_m = 1 + M_3(P^m)$, where $M_3(P^m)$ is the set of 3×3 matrices with entries in the ring P^m. Let $K_m = \tilde{K}_m \bigcap G$. Set

$$d_m = \begin{pmatrix} \varpi_E^{-2m} & & \\ & 1 & \\ & & \bar{\varpi}_E^{2m} \end{pmatrix}.$$

Let $J_m = d_m K_m d_m^{-1}$. Let $\hat{B}$ be the subgroup of lower triangular matrices in G, and $\hat{U} \subset \hat{B}$ be the subgroup of lower unipotent matrices in G. It is clear that any matrix in $\tilde{K}_m$ (hence in K_m and J_m) can be decomposed uniquely into a product of an upper unipotent matrix and a lower triangular matrix. The following Lemma is a consequence of the formulas for this decomposition.

LEMMA 5.1 ([Rod], Lemma 1):

$$K_m = (U \bigcap K_m)(\hat{B} \bigcap K_m) \quad \text{and} \quad J_m = (U \bigcap J_m)(\hat{B} \bigcap J_m)$$

with uniqueness of expression.

Let τ_m be a character of K_m defined by

$$\tau_m(k) = \psi_E(\varpi_E^{-2m} k_{1,2}) \quad \text{for all } k = (k_{i,l}) \in K_m.$$

It is easy to see that τ_m is indeed a character. We define characters ψ_m on J_m by

$$\psi_m(j) = \tau_m(d_m^{-1} j d_m), \quad j \in J_m.$$

It is easy to see that

$$\psi_m(j) = \psi_E(j_{1,2}) \quad \text{for all } j = (j_{i,l}) \in J_m.$$

Hence ψ_m and ψ agree on $J_m \bigcap U$.

Let (π, V) be a generic representation of G with Whittaker model $\mathcal{W}(\pi, \psi)$. Let $v \in V$ be such that $l(v) = W_v(1) = 1$, where l is a Whittaker functional associated to (π, V) (cf. end of Section 1). Let $U_m = J_m \bigcap U$ and du a Haar measure on U. For $m \geq 1$ define

$$v_m = \mathrm{vol}(U_m)^{-1} \int_{U_m} \psi(u)^{-1} \pi(u) v du. \tag{5.3}$$

LEMMA 5.2: *Let N be such that $\pi(K_N)v = v$. Then we have:*

(1) $W_{v_m}(1) = 1$.

(2) *For $m \geq N$, $\pi(j)v_m = \psi_m(j)v_m$ for all $j \in J_m$.*

(3) *If $k \geq m$ then $v_m = vol(U_m)^{-1} \int_{U_m} \psi(u)^{-1}\pi(u)v_k du$.*

Proof: (1) and (3) are clear. To prove (2), let $d\hat{b}$ be a right invariant Haar measure on $\hat{B}$. Using Lemma 5.1, we can see that $dj = d\hat{b}du$ is a Haar measure on J_m. For $m \geq N$ define

$$\tilde{v}_m = \mathrm{vol}(J_m)^{-1} \int_{J_m} \psi_m(j)^{-1} \pi(j) v du.$$

It is clear that $\tilde{v}_m$ satisfies (2). Since $\pi(K_m)v = v$ for $m \geq N$, and since $\hat{B} \bigcap J_m \subset K_m$, we have that, for $m \geq N$,

$$\begin{aligned} \tilde{v}_m &= \mathrm{vol}(U_m)^{-1} \mathrm{vol}(\hat{B} \bigcap J_m)^{-1} \int_{U_m} \int_{\hat{B} \bigcap J_m} \psi_m(u)^{-1} \pi(u) \pi(\hat{b}) v d\hat{b} du \\ &= \mathrm{vol}(U_m)^{-1} \int_{U_m} \psi(u)^{-1} \pi(u) v du = v_m. \qquad \blacksquare \end{aligned}$$

We call $v_m, m \geq N$ the Howe vectors associated to v. We suspect that v_m are independent of v and are determined uniquely by properties (1) and (2) of Lemma 5.2. This is true for GL_n [H]. However, we only need their existence for our purpose and the proof of existence (Lemma 5.2) generalizes to a large class of quasi-split reductive groups [B].

LEMMA 5.4: *Let $v_m, m \geq N$, be Howe vectors as in Lemma 5.2 and let W_{v_m} be the Whittaker functions associated to v_m. Then*

$$W_{v_m}(t(a)) = \begin{cases} 1 & \text{if } a \in 1 + P^m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: If $|y| \leq q^m$ then $u(y, -y\bar{y}/2) \in J_m$ and we have

$$u(ay, a\bar{a}y\bar{y}/2)t(a) = t(a)u(y, -y\bar{y}/2).$$

Thus we have

$$\psi(ay)W_{v_m}(t(a)) = W_{v_m}(t(a)u(y, -y\bar{y}/2)) = \psi(y)W_{v_m}(t(a)).$$

Since ψ has conductor R_E we get that $W_{v_m}(t(a)) = 0$ if $a \notin 1+P^m$. If $a \in 1+P^m$ then $t(a) \in J_m$ and our result follows from Lemma 5.2(2). ∎

Let

$$\hat{u}(y,z) = \begin{pmatrix} 1 & & \\ y & 1 & \\ z & -\bar{y} & 1 \end{pmatrix}, \quad y, z \in E, \quad z + \bar{z} = -y\bar{y},$$

and let

$$w = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Let $u(y, z)$ be as in Section 1. Then we have

$$(5.5) \qquad u(y,z)\hat{u}(\bar{y}/\bar{z}, 1/\bar{z}) = w \cdot \operatorname{diag}(1/\bar{z}, -\bar{z}/z, z)u(-y\bar{z}/z, \bar{z}).$$

LEMMA 5.6: *If $|z| \geq q^{6m}$ then $\hat{u}(\bar{y}/\bar{z}, 1/\bar{z}) \in J_m$.*

Proof: We have $\bar{y}/\bar{z} = -(1 + z/\bar{z})/y$, so $|\bar{y}/\bar{z}| \leq |y|^{-1}$. If $|y| \geq q^{3m}$ then we are done, and if $|y| < q^{3m}$ then $|\bar{y}/\bar{z}| = |y|/|z| < q^{-3m}$ and we are also done. ∎

The following Proposition characterizes the behavior of the Whittaker functions associated to Howe vectors on the open Bruhat cell. In particular we see that these Whittaker functions are determined by their values on elements of the form $t(a)w, a \in E^*$, and by the central character of π.

PROPOSITION 5.7: *Let (π, V) and (π', V') be generic representations of G with the same central character and define $v_m \in V$ and $v'_m \in V'$ as in Lemma 5.2. Let N be such an integer that v_m and v'_m are Howe vectors for $m \geq N$. Let $m \geq 3N$ and $u' \in U$. Then*

(a) $W_{v_m}(b) = W_{v'_m}(b)$ *for all* $b \in B$.

(b) *If* $u' \in U_m$ *then* $W_{v_m}(twu') = \psi(u')W_{v_m}(tw)$ *for all* $t \in T$.

(c) *If* $u' \notin U_m$ *then* $W_{v_m}(twu') = W_{v'_m}(twu')$ *for all* $t \in T$.

Proof: (a) Every element $b \in B$ can be written in the form $b = u_1 \cdot c \cdot t(a)$, where $u_1 \in U$, and $c \in C$, the center of G. Now we can use the fact that π and π' have the same central character and Lemma 5.4 to prove (a).

(b) follows from (2) of Lemma 5.2.

To prove (c), assume $u' = u(y', z') \notin J_m$. We have two cases. First assume that $|y'| > q^m$. Since $z' + \bar{z}' = -y'\bar{y}'$ we have that $|z'| > q^{2m}$. Since $m > N$, it follows from Lemma 5.2(3) that

$$W_{v_m}(twu') = \mathrm{vol}(U_m)^{-1} \int_{U_m} W_{v_N}(twu'u)\psi^{-1}(u)du. \tag{5.8}$$

For $u = u(y, z) \in U_m$, let $\tilde{u} = u'u$ where $\tilde{u} = u(\tilde{y}, \tilde{z})$. We have $\tilde{y} = y + y'$. Since $u \in U_m$, we have $|y| \le q^m$, hence $|\tilde{y}| = |y'| > q^m$. From the same argument as above we have that $|\tilde{z}| > q^{2m} \ge q^{6N}$. Thus, using Lemma 5.6 and relation (5.5), we can attach to $\tilde{u}$ a lower unipotent $j \in J_N$ such that $w\tilde{u}j = b \in B$. We get

$$W_{v_N}(twu'u) = W_{v_N}(tw\tilde{u}j) = W_{v_N}(tb) = W_{v'_N}(tb) = W_{v'_N}(tw\tilde{u}j) = W_{v'_N}(twu'u).$$

It follows from (5.8) that

$$W_{v_m}(twu') = W_{v'_m}(twu') \quad \text{for all } t \in T_H.$$

For the second case, we have $|y'| \le q^m$ and $|z'| > q^{3m}$. Looking again at (5.8) and at $\tilde{u} = u'u$, we can see that $\tilde{z} = z' - y'\bar{y} + z$, hence $|\tilde{z}| > q^{3m} > q^{6N}$. Repeating the argument above we conclude that

$$W_{v_m}(twu') = W_{v'_m}(twu') \quad \text{for all } t \in T_H. \qquad \blacksquare$$

6. Properties of the gamma factor

This section contains our main result, Theorem 6.3. The analogues for GL_n were proved by [J,S] and [He] using the Kirillov model for generic representations of GL_n. Our proof uses the properties of Howe vectors which we described above. We hope that this method will be amenable to generalizations for other groups. We have obtained similar results for GSp(4) [B].

For a set $A \subset F$ we denote by Υ_A the characteristic function of A. Let

$$\Phi_{i,l}(x,y) = \Upsilon_{P_F^i}(x)\Upsilon_{1+P_F^l}(y).$$

A simple computation shows that

$$\hat{\Phi}_{i,l}(x,y) = q_F^{-i-l}\psi(-x)\Upsilon_{P_F^{-l}}(x)\Upsilon_{P_F^{-i}}(y).$$

Let $r \in E$ be of the form $r = z\sqrt{\epsilon}, z \in F$. Then

$$\hat{x}(r) = m(d(\sqrt{\epsilon})) \cdot m\left[\begin{pmatrix} 1 & \\ z\epsilon & 1 \end{pmatrix}\right] \cdot m(d(\sqrt{\epsilon}^{-1})),$$
$$w \cdot x(r) = t(\sqrt{\epsilon}^{-1}) \cdot m(d(\sqrt{\epsilon})) \cdot m\left[\begin{pmatrix} 0 & 1 \\ -1 & -z \end{pmatrix}\right] \cdot m(d(\sqrt{\epsilon}^{-1})).$$

Let χ be a quasi-character of E^*, and $l = l(\chi)$ (cf. Section 1). Then

$$\begin{aligned} f(\hat{x}(r), \Phi_{i,l}, \chi, s) &= \int_{F^*} \Phi_{i,l}(z\epsilon y, y)\chi(y)|y|^s d^*y \\ &= \begin{cases} q_F^{-l} & \text{if } |z|_F \le q_F^{-i}|\epsilon|_F^{-1}, \\ 0 & \text{otherwise;} \end{cases} \end{aligned} \tag{6.1}$$

$$\begin{aligned} f(wx(r), \hat{\Phi}_{i,l}, \bar{\chi}^{-1}, 1-s) &= \chi(-\sqrt{\epsilon}) \int_{F^*} \hat{\Phi}(-y, -yz)\chi^{-1}(y)|y|^{1-s}d^*y \\ &= \begin{pmatrix} \chi(-\sqrt{\epsilon})q_F^{-i-l}\int_{|y|_F \le q_F^l} \psi_F(y)\chi^{-1}(y)|y|^{1-s}d^*y & \text{if } |z|_F \le q_F^{i-l}, \\ \chi(-\sqrt{\epsilon})q_F^{-i-l}\int_{|y|_F \le q^i|z|_F} \psi_F(y)\chi^{-1}(y)|y|^{1-s}d^*y & \text{if } |z| > q_F^{i-l}. \end{pmatrix} \end{aligned} \tag{6.2}$$

The equality in (6.2) is in the sense of analytic continuation. Let

$$c(s,\chi,\psi) = \int_{|y| \le q_F^{l(\chi)}} \chi(y)\psi_F(y)|y|^s d^*y$$

where χ is a quasi-character of F^* and ψ_F is a character of F trivial on R_F and nontrivial on $\varpi_F^{-1}R_F$. This integral converges absolutely for Re(s) large and can be meromorphically continued. If $l(\chi) > 0$ then $c(s,\chi,\psi)$ is nonzero, and if $l(\chi) = 0$ then $c(s,\chi,\psi)$ does not have a zero or a pole outside a finite number of values of q_F^s. (See, for example, [T] Lemma 5.1 and Theorem 5.5.)

We now prove our main Theorem of this paper.

THEOREM 6.3: *Let (π, V) and (π', V') be generic representations of G with the same central character.*

(1) *If $\gamma(s, \pi, \chi, \psi_F, \psi_E) = \gamma(s, \pi', \chi, \psi_F, \psi_E)$ for all quasi-characters χ of E^*, then π is isomorphic to π'.*
(2) *If χ is highly ramified, then $\gamma(s, \pi, \chi, \psi_F, \psi_E) = \gamma(s, \pi', \chi, \psi_F, \psi_E)$.*

A result similar to (1) was proved by Gelbart, Soudry and Rogawski [G,R,S]. They used global methods, i.e. the trace formula and theta correspondence.

Proof: By Lemma 4.9 and Lemma 4.10, it is enough to prove Theorem 6.3 for the characters ψ_F and ψ_E that we have fixed in the beginning of Section 5. Let $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\pi', \psi)$ be Whittaker models for π and π' respectively. Choose $v \in V$ and $v' \in V'$ such that $W_v(1) = W_{v'}(1) = 1$, $W_v \in \mathcal{W}(\pi, \psi)$, $W_{v'} \in \mathcal{W}(\pi', \psi)$. Let N_1 be an integer such that $\pi(K_{N_1})v = v$ and $\pi'(K_{N_1})v' = v'$. Let $N = 3N_1$ and $v_m \in V, v'_m \in V', m \geq N$ be Howe vectors associated with v and v' respectively, defined as in (5.3). Let $W_{v_m} \in \mathcal{W}(\pi, \psi)$, $W_{v'_m} \in \mathcal{W}(\pi', \psi)$ be the Whittaker functions associated with v_m and $v_{m'}$. By Proposition 5.7(a), we have $W_{v_m}(b) = W_{v'_m}(b)$ for all $b \in B$.

Let dz be the normalized Haar measure on F extended to

$$U_H = \{x(z\sqrt{\epsilon}), z \in F\} \cong F,$$

and $\hat{dz}$ be the normalized Haar measure on F extended to

$$\hat{U}_H = \{\hat{x}(z\sqrt{\epsilon}), z \in F\} \cong F.$$

We shall decompose the Haar measure on H, hence the H invariant measure on $U_H \setminus H$, in two ways. We shall use the first for the integral defining $Z(s, W, \Phi, \chi)$ and the second for the integral defining $Z(1-s, W, \hat{\Phi}, \bar{\chi}^{-1})$. We first look at the open dense set of H consisting of elements

$$x(y\sqrt{\epsilon})t(a)\hat{x}(z\sqrt{\epsilon}), \quad y, z \in F, \quad a \in E^*.$$

The Haar measure will be $|a|^{-1}dyd^*a\hat{dz}$, where d^*a is the Haar measure on E^*. The quotient measure will be $|a|^{-1}d^*a\hat{dz}$.

For the second decomposition we look at the open dense set of H consisting of elements

$$x(y\sqrt{\epsilon})t(a)wx(z\sqrt{\epsilon}), \quad y, z \in F, \quad a \in E^*.$$

The Haar measure will be $|a|^{-1}dyd^*adz$, and the quotient measure will be $|a|^{-1}d^*adz$.

Let χ be a quasi-character of E^* and let $l = l(\chi)$. Let $m \geq N$ and $m > l(\chi)$. We have

$$\begin{aligned} Z(s, W_{v_m}, \Phi_{i,l}, \chi) &= \int_{U_H \setminus H} W_{v_m}(h) f(s, h, \Phi_{i,l}, \chi) dh \\ &= \int_{E^*} \int_F W[t(a)\hat{x}(z\sqrt{\epsilon})]\chi(a)|a|^{s-1} f(s, \hat{x}(z\sqrt{\epsilon}), \Phi_{i,l}, \chi) dz d^*a. \end{aligned}$$

Using (6.1) this integral equals

$$q_F^{-l} \int_{E^*} \int_{|z|_F \leq q_F^{-i}|\epsilon|^{-1}} W_{v_m}[t(a)\hat{x}(z\sqrt{\epsilon})]\chi(a)|a|^{s-1} dz d^*a. \tag{6.4}$$

If i is large enough, $|z| \leq q_F^i |\epsilon|_F^{-1}$ implies $\hat{x}(z\sqrt{\epsilon}) \in J_m$. Hence, for such z

$$W_{v_m}[g\hat{x}(z\sqrt{\epsilon})] = W_{v_m}(g) \quad \text{for all } g \in G$$

(cf. Lemma 5.2 (2)). Hence (6.4) equals

$$q_F^{-l} \mathrm{vol}(\{\hat{x}(z\sqrt{\epsilon})\colon |z| \leq q_F^{-i}\}) \int_{E^*} W_{v_m}[t(a)]\chi(a)|a|^s d^*a.$$

Let $\alpha_i = \mathrm{vol}(\{\hat{x}(z\sqrt{\epsilon})\colon |z| \leq q_F^{-i}|\epsilon|_F^{-1}\})$. Using Lemma 5.4 we get that

$$Z(s, W_{v_m}, \Phi_{i,l}, \chi) = \alpha_i q_F^{-l} q_E^{-m}. \tag{6.5}$$

The same argument gives

$$Z(s, W_{v'_m}, \Phi_{i,l}, \chi) = \alpha_i q_F^{-l} q_E^{-m}. \tag{6.6}$$

Notice that we have proved here that $Z(s, W, \Phi, \chi)$ is not identically zero for every χ and s which concludes the proof of Theorem 4.7.

We now use the functional equation in Corollary 4.8 together with (6.5) and (6.6) to get

$$\begin{aligned} &\alpha_i q_F^{-l} q_E^{-m}[\gamma(s, \pi, \chi, \psi_F, \psi_E) - \gamma(s, \pi', \chi, \psi_F, \psi_E)] \\ &= Z(1-s, W_{v_m}, \hat{\Phi}_{i,l}, \bar{\chi}^{-1}) - Z(1-s, W_{v'_m}, \hat{\Phi}_{i,l}, \bar{\chi}^{-1}). \end{aligned} \tag{6.7}$$

Fix l as above. Let i be large enough such that $|z\sqrt{\epsilon}|_E \le q_E^{3m}$ implies $|z|_F \le q_F^{i-l}$. Using (6.2) we get

$$\begin{aligned}
&Z(1-s, W_{v_m}, \hat{\Phi}_{i,l}, \bar{\chi}^{-1}) - Z(1-s, W_{v'_m}, \hat{\Phi}_{i,l}, \bar{\chi}^{-1}) \\
&= \chi(-\sqrt{\epsilon}) q_F^{-i-l} c(1-s, \chi^{-1}, \psi_F) \\
(6.8)\cdot &\int\int_{|z\sqrt{\epsilon}|_E \le q_E^{3m}} [W_{v_m}(t(a)wx(\sqrt{\epsilon}z)) - W_{v'_m}(t(a)wx(\sqrt{\epsilon}z))]\chi^{-1}(\bar{a})|a|^{-s} dz d^*a \\
&+ \int\int_{|z\sqrt{\epsilon}|_E > q_E^{3m}} (W_{v_m}[t(a)wx(z)] - W_{v'_m}[t(a)wx(z)]) \\
(6.9) \quad &\cdot f(1-s, t(a)wx(z), \hat{\Phi}_{i,l}, \bar{\chi}^{-1})|a|^{-1} dz d^*a.
\end{aligned}$$

Using Proposition 5.7 (c) we have that (6.9) vanishes. Using Proposition 5.7 (b), we can see that (6.8) equals

$$\beta_{s,x} \int [W_{v_m}(t(a)w) - W_{v'_m}(t(a)w)]\chi^{-1}(\bar{a})|a|^{1-s} d^*a \tag{6.10}$$

where

$$\beta_{s,x} = \chi(-\sqrt{\epsilon}) q_F^{-i-l} c(1-s, \chi^{-1}, \psi_F) \mathrm{vol}\{x(z\sqrt{\epsilon}) \colon |z\sqrt{\epsilon}| \le q_E^{3m}\}.$$

By Lemma 5.2 (3) we can write

$$\begin{gathered}
W_{v_m}(t(a)w) - W_{v'_m}(t(a)w) = \\
\mathrm{vol}(U_m)^{-1} \int_{U_m} [W_{v_N}(t(a)wu) - W_{v'_N}(t(a)wu)]\psi^{-1}(u) du
\end{gathered}$$

and by Lemma 5.7 this integral equals

$$\mathrm{vol}(U_m)^{-1} \mathrm{vol}(U_N)[W_{v_N}(t(a)w) - W_{v'_N}(t(a)w)].$$

Using this, (6.10) and (6.7) we get

$$\begin{aligned}
&\lambda_{N,s}[\gamma(s, \pi, \chi, \psi_F, \psi_E) - \gamma(s, \pi', \chi, \psi_F, \psi_E)] \\
&= \int [W_{v_N}(t(a)w) - W_{v'_N}(t(a)w)]\chi^{-1}(\bar{a})|a|^{-s} d^*a
\end{aligned} \tag{6.11}$$

where $\lambda_{N,s} = \mathrm{vol}(U_m)\mathrm{vol}(U_N^{-1})\beta_{s,x}^{-1}\alpha_i q_F^{-l} q_E^{-m}$, and does not have a zero or a pole for all but finitely many values of q^s.

To prove (1) we notice that if $\gamma(s,\pi,\chi,\psi_F,\psi_E) = \gamma(s,\pi',\chi,\psi_F,\psi_E)$ for all quasi-characters χ of E^*, then we get

$$W_{v_N}(t(a)w) = W_{v'_N}(t(a)w) \quad \text{for all } a \in E^*. \tag{6.12}$$

This is because (6.11) is a power series in q^{-s} and the equality of the γ-factors implies that the coefficients of this power series vanish. (6.12) now follows from standard Fourier analysis (i.e., inverse Mellin transform). Another application of Proposition 5.7 gives us

$$W_{v_N}(g) = W_{v'_N}(g) \quad \text{for all } g \in G.$$

Since π and π' are irreducible and since the respective Whittaker models give realizations for π and π', we get that $\pi \cong \pi'$.

To prove (2) we notice that $W_{v_N}(t(a)w) - W_{v_N}(t(a)w)$ is a smooth function of a. So if χ is highly ramified then (6.11) vanishes and we get

$$\gamma(s,\pi,\chi,\psi_F,\psi_E) = \gamma(s,\pi',\chi,\psi_F,\psi_E). \qquad \blacksquare$$

7. A global application

In this section we give some indication on how to apply our local results to get a new proof of a strong multiplicity one theorem for the space of nondegenerate cuspidal automorphic representations of $U(2,1)$ (Theorem 7.2.13). A stronger version of this result can be obtained by using Theorem 13.3.5 in [Rog] and Theorem (c) in the introduction to [G,R,S]. Since we have not dealt at all with the archimedean primes we get a weaker version than what one can get from [Rog] and [G,R,S]. However, as we mentioned in Section 5, our methods seem amenable to generalizations.

Before stating this theorem we shall provide the global functional equation. This functional equation was not included in [G,PS,1].

7.1. Nondegenerate representations. Let F be a number field, E a quadratic extension of F and $\epsilon \in F$ such that $E = F[\sqrt{\epsilon}]$. Let $x \to \bar{x}$ be the nontrivial Galois automorphism of E over F, and $\mathbf{A}_F$, $\mathbf{A}_E$, $\mathbf{I}_F$, $\mathbf{I}_E$ the adeles and ideles of F and E respectively. We donte by $\mathbf{A} = \mathbf{A}_F$. Let ψ_F be a character of $\mathbf{A}_F/F$ and let ψ_E be a character of $\mathbf{A}_E/E$ defined by

$$\psi_E(x) = \psi_F\left(\frac{x-\bar{x}}{2\sqrt{\epsilon}}\right) \quad \text{for all } x \in \mathbf{A}_E.$$

We shall denote the standard absolute values on $\mathbf{A}_F$ and $\mathbf{A}_E$ by $||_F$ and $||_E$ respectively.

Let V be a 3-dimensional vector space over E with a Hermitian form given by the matrix J (cf. Section 1). Let $G = U(2,1)$ be the group of automorphisms of V preserving J. We shall use the same notation for the subgroups of G that were defined in Section 1. In particular, U is the unipotent radical of the upper triangular Borel subgroup of G. We let ψ be a character of the unipotent radical $U(\mathbf{A})$ obtained from ψ_E as in Section 1.

We define $U_H(\mathbf{A}_F)$ as in Section 1, i.e.,

$$U_H(\mathbf{A}_F) = \left\{ \begin{pmatrix} 1 & 0 & x \\ & 1 & 0 \\ & & 1 \end{pmatrix} \| x \in \mathbf{A}_E, x + \bar{x} = 0 \right\}.$$

Let (π, H_π) be an automorphic cuspidal representation of $G(\mathbf{A})$ which we assume is realized in $L_0^2(G(F)) \setminus G(\mathbf{A}))$. π is called hypercuspidal if for all $f \in H_\pi$ we have

$$\int_{U_H(F) \setminus U_H(\mathbf{A})} f(xg)dx = 0 \quad \text{for almost every } g \in G(\mathbf{A}).$$

We denote by $L_{0,1}^2$ the orthocomplement in L_0^2 of all hypercusp forms.

THEOREM 7.1.1 ([G,PS,1] Proposition 2.4):

(1) *$L_{0,1}^2$ has multiplicity 1.*

(2) *Each irreducible $(\pi, H_\pi) \subset L_{0,1}^2$ has nonvanishing Fourier coefficients along the standard maximal unipotent subgroup, hence a global Whittaker model.*

We call such (π, H_π) nondegenerate.

7.2. EISENSTEIN SERIES. The Eisenstein series appearing here are the ones alluded to in Remark 3.3 of [G,PS,1]. Notice that they are not the same as in (3.2.4) of [G,PS,1]. In studying them we follow [J] closely.

Let $S(\mathbf{A}_F^2)$ be the space of Schwartz functions on $\mathbf{A}_F^2$. For $\Phi \in S(\mathbf{A}_F^2)$ we define

$$\hat{\Phi}(x,y) = \int \Phi(u,v)\psi_F(yu - xv)dudv.$$

Let $\Phi \in S(F^2)$ and $s \in \mathbf{C}$ and χ be a quasi-character of $\mathbf{I}_F/F^*$. Let $g \in \mathrm{GL}_2(\mathbf{A}_F)$. We set

$$\tilde{f}(s,g,\Phi,\chi) = \int_{I_F} (g\Phi)(0,r)\chi(r)|r|_E^s d^*r. \tag{7.2.1}$$

This integral converges for $\mathrm{Re}(s)$ large and can be meromorphically continued to the whole complex plane [J].

Let

$$\tilde{E}(s,g,\Phi,\chi)=\sum_{\gamma}\tilde{f}(s,\gamma g,\Phi,\chi) \tag{7.2.2}$$

where the summation is taken over $B_{\mathrm{SL}}(F)\setminus \mathrm{SL}_2(F)$.

PROPOSITION 7.2.3 ([J] Proposition 19.3): *The series in* (7.2.2) *converges for* $\mathrm{Re}(s)$ *large and has meromorphic continuation to the whole plane. It satisfies the functional equation*

$$\tilde{E}(s,g,\Phi,\chi)=|\det(g)|_F^{1-2s}\chi^{-1}(\det(g))\tilde{E}(1-s,g,\hat{\Phi},\chi^{-1}). \tag{7.2.4}$$

Let H and S_H be subgroups of G as in Section 1. Every $h\in H(\mathbf{A}_F)$ can be written in the form

$$h=t(a)\cdot m(d(\sqrt{\epsilon}))\cdot h_1\cdot m(d(\sqrt{\epsilon}^{-1})) \tag{7.2.5}$$

where $a\in I_E$ and $h_1\in \mathrm{SL}_2(\mathbf{A}_F)$. This follows from the local decomposition (2.4) for the primes that do not split in E, and from decomposing the group $\mathrm{GL}_2(F_\nu)$ into a semi-direct product of $\mathrm{SL}_2(F_\nu)$ and F_ν^* when ν is a prime of F that splits in E. Notice that the decomposition (7.2.5) is not unique.

For $h\in H(\mathbf{A}_F)$ written in the form (7.2.5), $\Phi\in S(\mathbf{A}_F^2)$ and χ a quasi-character of $\mathbf{I}_{E^*}/E^*$ we set

$$f(s,h,\Phi,\chi)=\chi(a)|a|^s\tilde{f}(s,h_1\Phi,\chi).$$

It is easy to see that f is well defined (cf. Lemma 2.5). We let

$$E(s,h,\Phi,\chi)=\sum_{\gamma}f(s,\gamma h,\Phi,\chi) \tag{7.2.6}$$

where γ is in the set of representatives for $B_H(F)\setminus H(F)$.

PROPOSITION 7.2.7: *The series in* (7.2.6) *converges for* $\mathrm{Re}(s)$ *large and has meromorphic continuation to the whole plane. It satisfies the functional equation*

$$E(s,g,\Phi,\chi)=E(1-s,g,\hat{\Phi},\bar{\chi}^{-1}).$$

Proof: We let $e \cup \{wx(\sqrt{\epsilon}z)\colon z \in F\}$ be representatives of $B_H(F) \setminus H(F)$ where

$$w = \begin{pmatrix} & & \sqrt{\epsilon} \\ & 1 & \\ -\sqrt{\epsilon}^{-1} & & \end{pmatrix}.$$

Let $h \in H(\mathbf{A})$ and write $h = t(a) \cdot m(d(\sqrt{\epsilon})) \cdot h_1 \cdot m(d(\sqrt{\epsilon}^{-1}))$ as in (7.2.5). Let

$$\gamma = wx(\sqrt{\epsilon}z) \quad \text{and} \quad \tilde{\gamma} = \begin{pmatrix} 0 & 1 \\ -1 & -z \end{pmatrix}, \quad z \in F.$$

We have

$$wx(\sqrt{\epsilon}z)h = t(a) \cdot m(d(\sqrt{\epsilon})) \cdot m\begin{pmatrix} 0 & (a\bar{a})^{-1} \\ -a\bar{a} & -z \end{pmatrix} \cdot h_1 \cdot m(d(\sqrt{\epsilon}^{-1})).$$

It follows that

$$f(s, \gamma h, \Phi, \chi) = \chi(a)|a|^s f(s, \tilde{\gamma}, d(a\bar{a})h_1\Phi, \chi),$$

hence

$$E(s, h, \Phi, \chi) = \chi(a)|a|^s \tilde{E}(s, d(a\bar{a}), h_1\Phi, \chi). \tag{7.2.8}$$

The convergence of E now follows from the convergence of $\tilde{E}$. For the functional equation we have

$$\tilde{E}(s, d(a\bar{a}), h_1\Phi, \chi) = |a|^{1-2s}\chi^{-1}(a\bar{a})\tilde{E}(1-s, d(a\bar{a}), h_1\Phi, \bar{\chi}^{-1}). \tag{7.2.9}$$

Combining (7.2.8) and (7.2.9) we get the desired functional equation. ∎

Let (π, H_π) be a nondegenerate cuspidal automorphic representation of $G(\mathbf{A})$. For $f \in H_\pi$ we define

$$L(s, f, \Phi, \chi) = \int_{H(F) \setminus H(\mathbf{A})} f(h)E(s, h, \Phi, \chi)dh. \tag{7.2.10}$$

(See (3.2.1) and remark (3.3) in [G,PS,1].) Proposition 7.2.3 gives us the functional equation for $L(s, f, \Phi, \chi)$:

$$L(s, f, \Phi, \chi) = L(1-s, f, \hat{\Phi}, \bar{\chi}^{-1}). \tag{7.2.11}$$

Using proposition (3.5) in [G,PS,1] we get

$$L(s,f,\Phi,\chi)=\int_{U_H(\mathbf{A})\setminus H(\mathbf{A})} W_f^{\psi}(h)f(s,h,\Phi,\chi)dh \tag{7.2.12}$$

where

$$W_f^{\psi}(h)=\int_{U(F)\setminus U(\mathbf{A})} f(ug)\psi^{-1}(u)du.$$

Expressing the integral in (7.2.12) as an Euler product as in Proposition (3.6) of [G,PS,1] we get that $L(s,h,\Phi,\chi)$ is a product of the local zeta integrals defined in Section 2 (or a sum of such products).

Notice that by the results of Section 3, we can define local L-factors $L(s,\pi_\nu,\chi_v)$ for every non-archimedean place ν of F and a place v of E lying above ν. In what follows we shall assume that we can do so also for the archimedean places. Then we can define a global L-function $L(s,\pi,\chi)$. The functional equation for $L(s,\pi,\chi)$ (see [G,PS,1] (5.1)) will then come from (7.2.11), (7.2.12) and the local functional equation. We hope to do that in a future publication. (We need much less in order to prove the following result. All we need is (7.2.11), (7.2.12) and a nonvanishing result for the archimedean integrals.) The following Theorem is due to Gelbart, Rogawski and Soudry [Rog],[G,PS,1]. We bring here a new proof.

THEOREM 7.2.13: *Let $\pi=\bigotimes\pi_\nu$ and $\sigma=\bigotimes\sigma_\nu$ be cuspidal nondegenerate automorphic representations of $U(2,1)(\mathbf{A})$. Assume that there exist a finite set S of nonarchimidean valuations of F such that $\pi_\nu\cong\sigma_\nu$ for all $\nu\notin S$. Then $\pi=\sigma$.*

Proof: Let ν_0 be in S and let v_0 be a place of E lying above ν_0. For every quasi-character χ of E_{v_0} we can find a global character α of $\mathbf{I}_E/E^*$ such that α_v is highly ramified for any v lying above S, $v\neq v_0$ and such that $\alpha_{v_0}=\chi$ (lemma 12.5 in [J,L]).

Looking at the functional equations for $L(s,\pi,\alpha)$ and $L(s,\sigma,\alpha)$ and expressing these functional equations in the form of Euler products as in [G,PS,1], we can see that all but a finite number of Euler factors are the same, hence we get the following equality (see [Ca] or [G], Theorem 5.14 for a similar argument):

$$\prod_{\nu\in S}\frac{L(1-s,\tilde{\pi}_\nu,\bar{\alpha}_\nu^{-1})\epsilon(s,\pi_\nu,\alpha_\nu,\psi_\nu)}{L(s,\pi_\nu,\alpha_\nu)}=\prod_{\nu\in S}\frac{L(1-s,\tilde{\sigma}_\nu,\bar{\alpha}_v^{-1})\epsilon(s,\sigma_\nu,\alpha_\nu,\psi_\nu)}{L(s,\sigma_\nu,\alpha_\nu)}.$$

Here $\tilde{\pi}_\nu$ is the cotragredient of π_ν. This equality is the same as

$$\prod_{\nu\in S}\gamma(s,\pi_\nu,\alpha_\nu,\psi_\nu)=\prod_{\nu\in S}\gamma(s,\sigma_\nu,\alpha_\nu,\psi_\nu).$$

When ν splits in E, the local group is $\mathrm{GL}_3(F_\nu)$. It follows from ([G,PS,1] 3.7) that $\gamma(s,\pi_\nu,\alpha_\nu,\psi_\nu)$ is just the gamma factor attached to $\mathrm{GL}_3\times\mathrm{GL}_1$, and we have an analogous result to Theorem 6.3 (see [J,PS,S 1] Proposition 5.1 and Proposition 7.5.2). Since α_v is highly ramified except at v_0 we can use Theorem 6.3 (1) and its analogue for GL_3 to get

$$\gamma(s,\pi_{\nu_0},\chi,\psi_{\nu_0})=\gamma(s,\sigma_{\nu_0},\chi,\psi_{\nu_0})\qquad\text{for all quasi-characters } \chi \text{ of } E^*_{v_0},$$

where $\chi=\alpha_{v_0}$ is an arbitrarily chosen quasi-character of $E^*_{v_0}$. Using a density argument we can show that the central character of π is the same as the central character of σ, thus the central character of π_{ν_0} is the same as the central character of σ_{ν_0}. Hence, by Theorem 6.3 (2) or its analogue for GL_3, we have $\pi_{\nu_0}\cong\sigma_{\nu_0}$. Now we use the multiplicity one theorem in Theorem 7.1.1 to conclude that $\pi=\sigma$. ∎

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